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Exactly solvable one-dimensional model of resonance energy transfer

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Abstract. The dynamics of two atoms coupled to the vacuum radiation field is investigated within the framework of a one-dimensional model in the resonance dipole approximation. The exact solution of the resonance energy transfer problem is obtained. A many-atom generalization of the model is also discussed.

1. Introduction

In this article the problem of two two-level atoms coupled to the vacuum radiation field is studied within the framework of a one-dimensional (1D) model in the resonance dipole approximation. In the model, atoms interact only with photons propagating along the interatomic axis (x-axis), while interactions with all the other field harmonics are ignored. For instance, the model can be applied to atoms in a cylindrical cavity (waveguide) of a small radius, where due to the small radius, the characteristic frequencies of all transverse modes lie far from the frequency of the atomic transition under consideration.

The method developed here is based on an exact diagonalization of the model Hamiltonian, which allows us to study the temporal evolution of an arbitrary one-particle initial state of the atoms plus quantized field system. In particular, we derive explicit expressions for the temporal behaviour of the initial state of the system in which one of the atoms is excited, while the other and the field are in their ground states. This problem has been called the Fermi problem, because Fermi was first to examine a transfer of an excitation from one initially excited atom to the other [**?**]. The problem was the subject of many earlier investigations [2–9] and remains of interest at the present time [10–15]. In particular, the 1D model has been discussed previously by Arecci and Courtens [**?**] and Milonni and Knight [**?**]. In the next section of this article we discuss the details of our initial unnormalized model and its approximations.

In section 3, we generalize the standard Wigner–Weisskopf renormalization procedure to the case of two atoms, detail the necessary steps and obtain the model Hamiltonian that we shall treat. Then in section 4 we obtain the one-particle eigenstates.

One of the most fundamental questions related to this two-atom problem is the question of causality in the atom–atom interaction. In section 5, we show that within the framework of our model, we can give a rigorous proof that causality is not only preserved, but also

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that there exists no atom–atom correlation for $t < l$, where l is the interatomic separation. In section 6, we determine the spectral density of emitted photons.

We also apply our approach in section 7 to a many-atom 1D model and obtain the one-particle eigenstates of the model, which can be used in further studies of the dynamics of a one-particle excitation in a chain of atoms with arbitrary positions.

2. 1D two-atom unnormalized model

In order to keep the analysis simple, first we shall confine ourselves to the case of identical atoms, but the approach developed here can easily be applied to the more general case of two different atoms.

An interaction of two-level atoms with a 1D quantized electromagnetic field is described by the Hamiltonian

$$H = H_0 + V \tag{2.1}$$

$$H_0 = \omega_{12}(\sigma_1^z + \tfrac{1}{2}) + \omega_{12}(\sigma_2^z + \tfrac{1}{2}) + \int_0^\infty \frac{\mathrm{d}\omega}{2\pi}\omega[c_{\mathrm{R}}^+(\omega)c_{\mathrm{R}}(\omega) + c_{\mathrm{L}}^+(\omega)c_{\mathrm{L}}(\omega)] \tag{2.2}$$

$$V = \int_0^\infty \frac{\mathrm{d}\omega}{2\pi}\sqrt{\gamma(\omega)}[(c_{\mathrm{R}}(\omega)\mathrm{e}^{-\mathrm{i}\omega a} + c_{\mathrm{L}}(\omega)\mathrm{e}^{\mathrm{i}\omega a})\sigma_1^+ + (c_{\mathrm{R}}^+(\omega)\mathrm{e}^{\mathrm{i}\omega a} + c_{\mathrm{L}}^+(\omega)\mathrm{e}^{-\mathrm{i}\omega a})\sigma_1^-$$
$$+(c_{\mathrm{R}}(\omega)\mathrm{e}^{\mathrm{i}\omega a} + c_{\mathrm{L}}(\omega)\mathrm{e}^{-\mathrm{i}\omega a})\sigma_2^+ + (c_{\mathrm{R}}^+(\omega)\mathrm{e}^{-\mathrm{i}\omega a} + c_{\mathrm{L}}^+(\omega)\mathrm{e}^{\mathrm{i}\omega a})\sigma_2^-] \tag{2.3}$$

where $\gamma(\omega) = 2\pi\omega d^2/S_0$ and S_0 is the cross section of the waveguide. Here the two-level atoms with the transition frequency, ω_{12}, and the dipole moment of the transition, d, located at the points $\{x_1 = -a, x_2 = a\}$ are described by the spin operators $\sigma^i = (\sigma^x, \sigma^y, \sigma^z)$, $\sigma^\pm = \sigma^x \pm \mathrm{i}\sigma^y$ with the commutator

$$[\sigma_a^i, \sigma_b^j] = \mathrm{i}\delta_{ab}e^{ijk}\sigma^k \qquad i, j, k = x, y, z \qquad a, b = 1, 2 \tag{2.4}$$

where e^{ijk} is the unit antisymmetric tensor. The electric field is expanded in terms of transverse plane wave propagating along x-axis to positive (R) and negative (L) directions:

$$\boldsymbol{E}(x) = \boldsymbol{e}\int_0^\infty \frac{\mathrm{d}\omega}{2\pi}\sqrt{\frac{2\pi\omega}{S_0}}[c_{\mathrm{R}}(\omega)\mathrm{e}^{\mathrm{i}\omega x} + c_{\mathrm{L}}(\omega)\mathrm{e}^{-\mathrm{i}\omega x}] \tag{2.5}$$

where $\boldsymbol{e}$ is the polarization vector. The operators $c_\alpha(\omega)$, $\alpha = \mathrm{R}, \mathrm{L}$ obey the Bose commutation relations

$$[c_\alpha(\omega)c_\beta^+(\omega')] = 2\pi\delta_{\alpha\beta}\delta(\omega - \omega'). \tag{2.6}$$

An atom–field interaction is described in the dipole approximation by the operator

$$V = -[\hat{\boldsymbol{d}}_1\boldsymbol{E}(-a) + \hat{\boldsymbol{d}}_2\boldsymbol{E}(a)] \tag{2.7}$$

where the operator of the dipole moment of the resonance transition is given by

$$\hat{\boldsymbol{d}} = \boldsymbol{d}(\sigma_a^+ + \sigma_a^-). \tag{2.8}$$

In equation (2.3) we used the resonance (rotating-wave) approximation and omitted terms $c_\alpha^+\sigma_a^+$ $(c_\alpha\sigma_a^-)$ which describe a simultaneous creation (annihilation) of both a photon and an atomic excitation. Then, the operator

$$N = \sum_{a=1,2}(\sigma_a^z + \tfrac{1}{2}) + \sum_{\alpha=\mathrm{R},\mathrm{L}}\int_0^\infty \frac{\mathrm{d}\omega}{2\pi}c_\alpha^+(\omega)c_\alpha(\omega) \tag{2.9}$$

commutes with the Hamiltonian and should be treated as the excitation number operator of the atoms plus field system.

3. Renormalization procedure

Since the 'particle' number operator N commutes with the Hamiltonian, all the model eigenstates can be classified with respect to the number of particles, or eigenvalues of the operator N.

The vacuum state of the atoms plus field system, $|0\rangle$, contains no particle, $N|0\rangle = H|0\rangle = 0$, and is defined by

$$c_\alpha(\omega)|0\rangle = \sigma_a^-|0\rangle = 0. \tag{3.1}$$

Let us look for one-particle eigenstates, $N|\Psi_1\rangle = 1 \cdot |\Psi_1\rangle$, as a linear superposition of all the degrees of the freedom of the model,

$$|\Psi\rangle = \left[\xi_1\sigma_1^+ + \xi_2\sigma_2^+ + \int_0^\infty \frac{\mathrm{d}\omega}{2\pi}(f_\mathrm{R}(\omega)c_\mathrm{R}^+(\omega) + f_\mathrm{L}(\omega)c_\mathrm{L}^+(\omega))\right]|0\rangle. \tag{3.2}$$

Then, the Schrödinger equation $(H-\lambda)|\Psi>=0$ takes the form

$$\omega f_\mathrm{R}(\omega,\lambda) + \sqrt{\gamma(\omega)}(\xi_1(\lambda)\mathrm{e}^{\mathrm{i}\omega a} + \xi_2(\lambda)\mathrm{e}^{-\mathrm{i}\omega a}) = \lambda f_\mathrm{R}(\omega,\lambda) \tag{3.3}$$

$$\omega f_\mathrm{L}(\omega,\lambda) + \sqrt{\gamma(\omega)}(\xi_1(\lambda)\mathrm{e}^{-\mathrm{i}\omega a} + \xi_2(\lambda)\mathrm{e}^{\mathrm{i}\omega a}) = \lambda f_\mathrm{L}(\omega,\lambda) \tag{3.4}$$

$$\omega_{12}\xi_1(\lambda) + \int_0^\infty \frac{\mathrm{d}\omega}{2\pi}\sqrt{\gamma(\omega)}(f_\mathrm{R}(\omega,\lambda)\mathrm{e}^{-\mathrm{i}\omega a} + f_\mathrm{L}(\omega,\lambda)\mathrm{e}^{\mathrm{i}\omega a}) = \lambda\xi_1(\lambda) \tag{3.5}$$

$$\omega_{12}\xi_2(\lambda) + \int_0^\infty \frac{\mathrm{d}\omega}{2\pi}\sqrt{\gamma(\omega)}(f_\mathrm{R}(\omega,\lambda)\mathrm{e}^{\mathrm{i}\omega a} + f_\mathrm{L}(\omega,\lambda)\mathrm{e}^{-\mathrm{i}\omega a}) = \lambda\xi_2(\lambda). \tag{3.6}$$

The solutions of these equations are twice degenerate, i.e. there are two solutions (the symmetric solution $\xi_1 = \xi_2 = \xi$ and the antisymmetric one $\xi_1 = -\xi_2 = \xi$)) that correspond to the same eigenenergy, λ. To see that one must renormalize these eigenstates, let us consider, for instance, only the symmetric solution. We then have

$$\omega f(\omega,\lambda) + \sqrt{\gamma(\omega)}(\mathrm{e}^{\mathrm{i}\omega a} + \mathrm{e}^{-\mathrm{i}\omega a})\xi(\lambda) = \lambda f(\omega,\lambda) \tag{3.7}$$

$$\omega_{12}\xi + \int_0^\infty \frac{\mathrm{d}\omega}{2\pi}\sqrt{\gamma(\omega)}(\mathrm{e}^{-\mathrm{i}\omega a} + \mathrm{e}^{\mathrm{i}\omega a})f(\omega,\lambda) = \lambda\xi(\lambda) \tag{3.8}$$

where $f_\mathrm{R} = f_\mathrm{L} = f$. Substitution

$$f(\omega,\lambda) = (\mathrm{e}^{-\mathrm{i}\omega a} + e^{\mathrm{i}\omega a})\psi(\omega,\lambda)$$

yields

$$\omega\psi(\omega,\lambda) + \sqrt{\gamma(\omega)}\xi(\lambda) = \lambda\psi(\omega,\lambda) \tag{3.9}$$

$$(\lambda-\omega_{12})\xi(\lambda) = \int_0^\infty \frac{\mathrm{d}\omega}{2\pi}\sqrt{\gamma(\omega)}[2 + \mathrm{e}^{-\mathrm{i}\omega l} + \mathrm{e}^{\mathrm{i}\omega l}]\psi(\omega,\lambda) \tag{3.10}$$

where $l = 2a$ is the interatomic separation.

The general solution of equation (3.9) is obviously given by

$$\psi(\omega,\lambda) = 2\pi\delta(\lambda-\omega)\phi(\lambda) + \frac{\sqrt{\gamma(\omega)}}{\lambda-\omega+\mathrm{i}0}\xi(\lambda) \tag{3.11}$$

where $\phi(\lambda)$ is an arbitrary function. Substituting this expression into equation (3.10) we find

$$(\lambda-\omega_{12})\xi(\lambda) = \sqrt{\gamma(\lambda)}[2 + \mathrm{e}^{-\mathrm{i}\lambda l} + \mathrm{e}^{\mathrm{i}\lambda l}]\phi(\lambda) + \xi(\lambda)\int_0^\infty \frac{\mathrm{d}\omega}{2\pi}\frac{\gamma(\omega)}{\lambda-\omega+\mathrm{i}0}(2 + \mathrm{e}^{-\mathrm{i}\omega l} + \mathrm{e}^{\mathrm{i}\omega l}). \tag{3.12}$$

Since the function $\gamma(\omega) \propto \omega$, the integral diverges and must be renormalized. Within the framework of the resonance (rotating-wave) approximation, which we have already introduced in the previous section, one neglects the ω-dependence of $\gamma(\omega)$ and first replaces $\gamma(\omega) \to \gamma(\omega_{12}) =$ constant. Using the transform

$$\int_0^\infty = \int_{-\infty}^\infty - \int_{-\infty}^0$$

it is convenient now to rewrite the integral part of the right-hand side of equation (3.12) as follows:

$$I = I_1 - I_2 \tag{3.13}$$

$$I_1 = \gamma\xi(\lambda) \int_{-\infty}^\infty \frac{\mathrm{d}\omega}{2\pi} \frac{1}{\lambda - \omega + \mathrm{i}0} [2 + \mathrm{e}^{-\mathrm{i}\omega l} + \mathrm{e}^{\mathrm{i}\omega l}] \tag{3.14}$$

$$I_2 = \gamma\xi(\lambda) \int_0^\infty \frac{\mathrm{d}\omega}{2\pi} \frac{1}{\lambda + \omega} [2 + \mathrm{e}^{-\mathrm{i}\omega l} + \mathrm{e}^{\mathrm{i}\omega l}]. \tag{3.15}$$

The first integral, I_1, is completely defined and yields

$$I_1 = -\mathrm{i}\gamma\xi(\lambda)(1 + \mathrm{e}^{\mathrm{i}\lambda l}) \tag{3.16}$$

while the second one, I_2, diverges and has to be renormalized. It should be emphasized that this integral is purely real and contains only corrections to the transition frequency, ω_{12}, due to the atom–field coupling, which affects the time evolution of the system. However, these corrections will be unimportant for the causal behaviour of the system. Moreover, since the model under consideration is nonrelativistic and does not include the relativistic behaviour of atomic electrons, we will neglect the part independent of the interatomic separation, l, (Lamb shift) and simply take the transition frequency of a single atom, ω_{12}, is taken to be the experimental value. Then, we have

$$I_2^{(\mathrm{renor})} = 2\gamma\xi(\lambda) \int_0^\infty \frac{\mathrm{d}\omega}{2\pi} \frac{\cos(\omega l)}{\omega + \lambda} = 2\gamma\xi(\lambda) \int_0^\infty \frac{\mathrm{d}\omega}{2\pi} \frac{\cos(\omega l)}{\omega + \omega_{12}} \tag{3.17}$$

where, in the last step, we have replaced the λ-dependence of the integral with $\lambda \to \omega_{12}$. Thus, finally we find for $\xi(\lambda)$ in the symmetric eigenstate

$$\xi^{(\mathrm{s})}(\lambda) = \sqrt{\gamma} \frac{2 + \mathrm{e}^{-\mathrm{i}\lambda l} + \mathrm{e}^{\mathrm{i}\lambda l}}{\lambda - \omega_{12}^{(\mathrm{s})} + \mathrm{i}\gamma(1 + \mathrm{e}^{\mathrm{i}\lambda l})} \phi(\lambda) \tag{3.18}$$

where

$$\omega_{12}^{(\mathrm{s})} = \omega_{12} - \gamma \int_0^\infty \frac{\mathrm{d}\omega}{2\pi} \frac{\cos(\omega l)}{\omega + \omega_{12}} = \omega_{12} + \gamma[\sin(\omega_{12} l)\mathrm{si}(\omega_{12} l) + \cos(\omega_{12} l)\mathrm{ci}(\omega_{12} l)] \tag{3.19}$$

and the integral functions are defined as usual [16]:

$$\mathrm{si}(x) = -\int_x^\infty \mathrm{d}t \frac{\sin t}{t} \qquad \mathrm{ci}(x) = -\int_x^\infty \mathrm{d}t \frac{\cos t}{t}.$$

The second term in (3.19) describes a correction to the transition frequency due to an effective interatomic interaction created by an atom–field coupling. This term lifts the degeneracy of two independent atoms and is a function of the interatomic separation. For large interatomic separations, $l \gg \omega_{12}^{-1}$, its contribution falls away due to $\lim(x^a \mathrm{si}(x)) = \lim(x^a \mathrm{ci}(x)) = 0$, as $x \to \infty$ for $a < 1$, and can be omitted. On the other hand, for small separations, since $\mathrm{ci}(x) \sim \ln(x)$, this correction then becomes comparable to the transition frequency and, hence, the resonance approximation fails only at exponentially small, non-physical interatomic separations, $l \sim \omega_{12}^{-1} \exp(-\omega_{12}/\gamma)$. Thus, within the framework of

the 1D model, the resonance approximation remains adequate for any physical interatomic distances.

Let us now turn to the antisymmetric eigenstate ($\xi_1 = -\xi_2$). Here, instead of (3.19), one obtains

$$\omega_{12}^{(a)} = \omega_{12} + \gamma \int_0^\infty \frac{d\omega}{2\pi} \frac{\cos(\omega l)}{\omega + \omega_{12}}. \tag{3.20}$$

To keep the notation simple, for the moment we shall confine ourselves only to the case of long interatomic separations, $\omega_{12} l \gg 1$, where the corrections to the transition frequency given by (3.19) and (3.20) can be neglected. These corrections will be restored into the model eigenstates at the end of the next section.

After all the above is done, one can easily see that the renormalization procedure used here is equivalent to simply first introducing an effective model Hamiltonian and starting from there. Namely, to renormalized our starting model, one replaces $\gamma(\omega) \to \gamma = \text{constant}$ in (2.2) and (2.3), and extends the lower limit of integration over the frequency down to $-\infty$. Then, in terms of operators

$$\epsilon_R(x) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} c_R(\omega) e^{i\omega x} \qquad \epsilon_L(x) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} c_L(\omega) e^{-i\omega x} \tag{3.21}$$

the effective Hamiltonian takes the form

$$H = H_0 + V \tag{3.22}$$

$$H_0 = \omega_{12}(\sigma_1^z + \sigma_2^z + 1) - i \int_{-\infty}^{\infty} dx \left(\epsilon_R^+(x) \frac{\partial}{\partial x} \epsilon_R(x) - \epsilon_L^+(x) \frac{\partial}{\partial x} \epsilon_L(x) \right) \tag{3.23}$$

$$V = \sqrt{\gamma} \sum_{a=1,2} \sum_{\alpha=R,L} \int_{-\infty}^{\infty} dx\, \delta(x - x_a)(\epsilon_\alpha(x)\sigma_a^+ + \epsilon_\alpha^+(x)\sigma_a^-) \tag{3.24}$$

while the one-particle eigenstates should be rewritten as follows:

$$|\Psi\rangle = \left[\xi_1 \sigma_1^+ + \xi_2 \sigma_2^+ + \int_{-\infty}^{\infty} dx (f_R(x)\epsilon_R^+(x) + f_L(x)\epsilon_L^+(x)) \right] |0\rangle. \tag{3.25}$$

One should note that the renormalization procedure used here is simply a generalization of the standard Wigner–Weisskopf approximation [17] for the case of two atoms. Lastly, we again note that in the case of short interatomic separations, $\omega_{12} l < 1$, the atomic transition frequency will be given by the expressions (3.19) and (3.20) for symmetric and antisymmetric eigenstates, respectively.

4. One-particle eigenstates

In terms of the effective Hamiltonian (3.22)–(3.24) the Schrödinger equation, $(H - \lambda)|\Psi_1\rangle = 0$, takes the following form:

$$-i \frac{d}{dx} f_R(x) + \sqrt{\gamma} \sum_{a=1,2} \xi_a \delta(x - x_a) = \lambda f_R(x) \tag{4.1}$$

$$i \frac{d}{dx} f_L(x) + \sqrt{\gamma} \sum_{a=1,2} \xi_a \delta(x - x_a) = \lambda f_L(x) \tag{4.2}$$

$$\omega_{12} \xi_a + \sqrt{\gamma} \int_{-\infty}^{\infty} dx\, \delta(x - x_a)(f_R(x) + f_L(x)) = \lambda \xi_a. \tag{4.3}$$

Substituting

$$f_R(x) = e^{i\lambda x} \phi_R(x) \qquad f_L(x) = e^{-i\lambda x} \phi_L(x)$$

yields

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}\phi_{\mathrm{R}}(x) = \sqrt{\gamma}[\xi_1 \mathrm{e}^{-\mathrm{i}\lambda x_1}\delta(x-x_1) + \xi_2 \mathrm{e}^{-\mathrm{i}\lambda x_2}\delta(x-x_2)] \tag{4.4}$$

$$-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}\phi_{\mathrm{L}}(x) = \sqrt{\gamma}\left[\xi_1 \mathrm{e}^{\mathrm{i}\lambda x_1}\delta(x-x_1) + \xi_1 \mathrm{e}^{\mathrm{i}\lambda x_2}\delta(x-x_2)\right] \tag{4.5}$$

$$(\lambda - \omega_{12})\xi_a = \sqrt{\gamma}\int_{-\infty}^{\infty}\mathrm{d}x\,\delta(x-x_a)[\phi_{\mathrm{R}}(x)\mathrm{e}^{\mathrm{i}\lambda x_a} + \phi_{\mathrm{L}}(x)\mathrm{e}^{-\mathrm{i}\lambda x_a}]. \tag{4.6}$$

The general solution of the equations (4.4) and (4.5) is

$$\phi_{\mathrm{R}}(x) = C_{\mathrm{R}} - \mathrm{i}\sqrt{\gamma}[\xi_1 \mathrm{e}^{-\mathrm{i}\lambda x_1}\theta(x-x_1) + \xi_2 \mathrm{e}^{-\mathrm{i}\lambda x_2}\theta(x-x_2)] \tag{4.7}$$

$$\phi_{\mathrm{L}}(x) = C_{\mathrm{L}} - \mathrm{i}\sqrt{\gamma}[\xi_1 \mathrm{e}^{\mathrm{i}\lambda x_1}\theta(x_1-x) + \xi_2 \mathrm{e}^{\mathrm{i}\lambda x_2}\theta(x_2-x)] \tag{4.8}$$

where $C_{\mathrm{R,L}}$ are arbitrary constants. C_{R} represents the amplitude of the right-going photons as $x \to -\infty$ while C_{L} represents the amplitude of the left-going photons as $x \to +\infty$. Thus these two quantities give the initial input fluxes of photons from both directions.

Substituting (4.7) and (4.8) into (4.6), we obtain the following set of the algebraic equations for $\xi_{1,2}$:

$$(\lambda - \omega_{12} + \mathrm{i}\gamma)\xi_1 + \mathrm{i}\gamma g\xi_2 = \sqrt{\gamma}(C_{\mathrm{R}}\mathrm{e}^{\mathrm{i}\lambda x_1} + C_{\mathrm{L}}\mathrm{e}^{-\mathrm{i}\lambda x_1}) \tag{4.9}$$

$$\mathrm{i}\gamma g\xi_1 + (\lambda - \omega_{12} + \mathrm{i}\gamma)\xi_2 = \sqrt{\gamma}(C_{\mathrm{R}}\mathrm{e}^{\mathrm{i}\lambda x_2} + C_{\mathrm{L}}\mathrm{e}^{-\mathrm{i}\lambda x_2}) \tag{4.10}$$

where the integral $\int \mathrm{d}x\,\delta(x)\theta(x)$ has been taken to be $1/2$, and

$$g = \exp(\mathrm{i}\lambda|x_1 - x_2|). \tag{4.11}$$

In solving (4.9) and (4.10), it becomes convenient to use eigenstates of the parity operator. To find these states, we start by defining $\xi_\pm = \xi_1 \pm \xi_2$. Then (4.9) and (4.10) become

$$[\lambda - \omega_{12} + \mathrm{i}\gamma(1+g)]\xi_+ = \sqrt{\gamma}[C_{\mathrm{R}}(\mathrm{e}^{\mathrm{i}\lambda x_1} + \mathrm{e}^{\mathrm{i}\lambda x_2}) + C_{\mathrm{L}}(\mathrm{e}^{-\mathrm{i}\lambda x_1} + \mathrm{e}^{-\mathrm{i}\lambda x_2})] \tag{4.12}$$

$$[\lambda - \omega_{12} + \mathrm{i}\gamma(1-g)]\xi_- = \sqrt{\gamma}[C_{\mathrm{R}}(\mathrm{e}^{\mathrm{i}\lambda x_1} - \mathrm{e}^{\mathrm{i}\lambda x_2}) + C_{\mathrm{L}}(\mathrm{e}^{-\mathrm{i}\lambda x_1} - \mathrm{e}^{-\mathrm{i}\lambda x_2})]. \tag{4.13}$$

For the symmetric solution, ξ_- must vanish and (4.13) then gives

$$C_{\mathrm{R}}^{(\mathrm{s})} = \mathrm{e}^{-\mathrm{i}\lambda x_1} + \mathrm{e}^{-\mathrm{i}\lambda x_2} \qquad C_{\mathrm{L}}^{(\mathrm{s})} = \mathrm{e}^{\mathrm{i}\lambda x_1} + \mathrm{e}^{\mathrm{i}\lambda x_2} \tag{4.14}$$

which leads to the solution

$$\xi_1 = \xi_2 = \xi^{(\mathrm{s})}(\lambda) = \frac{2}{\sqrt{\gamma}}\frac{\Gamma^{(\mathrm{s})}}{\lambda - \Omega^{(\mathrm{s})} + \mathrm{i}\Gamma^{(\mathrm{s})}} \tag{4.15}$$

where

$$\Omega^{(\mathrm{s})} = \omega_{12} + \gamma g'' \qquad \Gamma^{(\mathrm{s})} = \gamma(1+g') \tag{4.16}$$

and g', g'' are the real and imaginary parts of the function g, $g = g' + \mathrm{i}g''$. Substituting (4.14) and (4.15) into (4.7) and (4.8) yields for the photon wavefunctions in the symmetric eigenstates

$$f_{\mathrm{R}}^{(\mathrm{s})}(x,\lambda) = \frac{\lambda - \Omega^{(\mathrm{s})} - \mathrm{i}\Gamma^{(\mathrm{s})}\mathrm{sgn}(x-x_1)}{\lambda - \Omega^{(\mathrm{s})} + \mathrm{i}\Gamma^{(\mathrm{s})}}\mathrm{e}^{\mathrm{i}\lambda(x-x_1)} + \frac{\lambda - \Omega^{(\mathrm{s})} - \mathrm{i}\Gamma^{(\mathrm{s})}\mathrm{sgn}(x-x_2)}{\lambda - \Omega^{(\mathrm{s})} + \mathrm{i}\Gamma^{(\mathrm{s})}}\mathrm{e}^{\mathrm{i}\lambda(x-x_2)} \tag{4.17}$$

$$f_{\mathrm{L}}^{(\mathrm{s})}(x,\lambda) = \frac{\lambda - \Omega^{(\mathrm{s})} - \mathrm{i}\Gamma^{(\mathrm{s})}\mathrm{sgn}(x_1-x)}{\lambda - \Omega^{(\mathrm{s})} + \mathrm{i}\Gamma^{(\mathrm{s})}}\mathrm{e}^{-\mathrm{i}\lambda(x-x_1)} + \frac{\lambda - \Omega^{(\mathrm{s})} - \mathrm{i}\Gamma^{(\mathrm{s})}\mathrm{sgn}(x_2-x)}{\lambda - \Omega^{(\mathrm{s})} + \mathrm{i}\Gamma^{(\mathrm{s})}}\mathrm{e}^{-\mathrm{i}\lambda(x-x_2)} \tag{4.18}$$

where

$$\mathrm{sgn}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0. \end{cases}$$

Correspondingly, for the antisymmetric solution, ξ_+ must vanish, giving

$$C_{\mathrm{R}}^{(\mathrm{a})} = \mathrm{e}^{-\mathrm{i}\lambda x_1} - \mathrm{e}^{-\mathrm{i}\lambda x_2} \qquad C_{\mathrm{L}}^{(\mathrm{a})} = \mathrm{e}^{\mathrm{i}\lambda x_1} - \mathrm{e}^{\mathrm{i}\lambda x_2} \tag{4.19}$$

which leads to the antisymmetric solution

$$\xi_1 = -\xi_2 = \xi^{(\mathrm{a})}(\lambda) = \frac{2}{\sqrt{\gamma}} \frac{\Gamma^{(\mathrm{a})}}{\lambda - \Omega^{(\mathrm{a})} + \mathrm{i}\Gamma^{(\mathrm{a})}} \tag{4.20}$$

and

$$f_{\mathrm{R}}^{(\mathrm{a})}(x,\lambda) = \frac{\lambda - \Omega^{(\mathrm{a})} - \mathrm{i}\Gamma^{(\mathrm{a})}\mathrm{sgn}(x - x_1)}{\lambda - \Omega^{(\mathrm{a})} + \mathrm{i}\Gamma^{(\mathrm{a})}} \mathrm{e}^{\mathrm{i}\lambda(x-x_1)} - \frac{\lambda - \Omega^{(\mathrm{a})} - \mathrm{i}\Gamma^{(\mathrm{a})}\mathrm{sgn}(x - x_2)}{\lambda - \Omega^{(\mathrm{a})} + \mathrm{i}\Gamma^{(\mathrm{a})}} \mathrm{e}^{\mathrm{i}\lambda(x-x_2)} \tag{4.21}$$

$$f_{\mathrm{L}}^{(\mathrm{a})}(x,\lambda) = \frac{\lambda - \Omega^{(\mathrm{a})} - \mathrm{i}\Gamma^{(\mathrm{a})}\mathrm{sgn}(x_1 - x)}{\lambda - \Omega^{(\mathrm{a})} + \mathrm{i}\Gamma^{(\mathrm{a})}} \mathrm{e}^{-\mathrm{i}\lambda(x-x_1)} - \frac{\lambda - \Omega^{(\mathrm{a})} - \mathrm{i}\Gamma^{(\mathrm{a})}\mathrm{sgn}(x_2 - x)}{\lambda - \Omega^{(\mathrm{a})} + \mathrm{i}\Gamma^{(\mathrm{a})}} \mathrm{e}^{-\mathrm{i}\lambda(x-x_2)} \tag{4.22}$$

where

$$\Omega^{(\mathrm{a})} = \omega_{12} - \gamma g'' \qquad \Gamma^{(\mathrm{a})} = \gamma(1 - g'). \tag{4.23}$$

Thus, we have two independent eigenstates of our model with eigenenergy λ. They are

$$|\lambda, s\rangle = \left[\xi^{(\mathrm{s})}(\lambda)(\sigma_1^+ + \sigma_2^+) + \int_{-\infty}^{\infty} \mathrm{d}x (f_{\mathrm{R}}^{(\mathrm{s})}(x,\lambda)\epsilon_{\mathrm{R}}^+(x) + f_{\mathrm{L}}^{(\mathrm{s})}(x,\lambda)\epsilon_{\mathrm{L}}^+(x))\right]|0\rangle \tag{4.24}$$

$$|\lambda, a\rangle = \left[\xi^{(\mathrm{a})}(\lambda)(\sigma_1^+ - \sigma_2^+) + \int_{-\infty}^{\infty} \mathrm{d}x (f_{\mathrm{R}}^{(\mathrm{a})}(x,\lambda)\epsilon_{\mathrm{R}}^+(x) + f_{\mathrm{L}}^{(\mathrm{a})}(x,\lambda)\epsilon_{\mathrm{L}}^+(x))\right]|0\rangle \tag{4.25}$$

which form the orthogonal basis, $\langle \mu, s|\lambda, a\rangle = 0$, in the two-dimensional space of the model eigenstates:

$$\langle \mu, s|\lambda, s\rangle = \frac{8\pi}{\gamma}\Gamma^{(\mathrm{s})}\delta(\lambda - \mu) \qquad \langle \mu, a|\lambda, a\rangle = \frac{8\pi}{\gamma}\Gamma^{(\mathrm{a})}\delta(\lambda - \mu). \tag{4.26}$$

These results can easily be extended to the case of short interatomic separations, $\omega_{12} l < 1$. To do it one simply needs to redefine the quantities $\Omega^{(\mathrm{s,a})}$ as follows:

$$\Omega^{(\mathrm{s})} = \omega_{12}^{(\mathrm{s})} + \gamma g'' \qquad \Omega^{(\mathrm{a})} = \omega_{12}^{(\mathrm{a})} - \gamma g'' \tag{4.27}$$

where $\omega_{12}^{(\mathrm{s,a})}$ are given by (3.19) and (3.20), respectively. Then, all the above expressions for the one-particle eigenstates become valid for arbitrary interatomic separations.

5. Fermi problem

Now we shall address the question of whether or not an excited atom can transmit a photon to a second atom, in the ground state, faster than the speed of light. We shall show that within the framework of the effective Hamiltonian and the generalized Wigner–Weisskopf renormalization procedure, causality is preserved.

At some initial moment of time $t = 0$, let one of the atoms, say atom number 1, be excited, while the second one and the field are in their ground states, i.e.

$$|\mathrm{In}\rangle = \sigma_1^+|0\rangle. \tag{5.1}$$

This state is a one-particle state $N|\mathrm{In}\rangle = 1 \cdot |\mathrm{In}\rangle$, and, hence, can be represented as a linear superposition of the one-particle eigenstates of the model,

$$|\mathrm{In}\rangle = \sum_{\sigma=\mathrm{s,a}} \int_{-\infty}^{\infty} \frac{\mathrm{d}\lambda}{2\pi} A^{(\sigma)}(\lambda)|\lambda, \sigma\rangle \tag{5.2}$$

where the coefficients are found from (5.1) and (4.26). They are

$$A^{(\sigma)}(\lambda) = \frac{\sqrt{\gamma}}{2} \frac{1}{\lambda - \Omega^{(\sigma)} - \mathrm{i}\Gamma^{(\sigma)}}. \tag{5.3}$$

The dynamics of the initial state is determined by the Hamiltonian

$$|\Phi(t)\rangle = \exp(-\mathrm{i}Ht)|\mathrm{In}\rangle = \sum_{\sigma=\mathrm{s,a}} \int_{-\infty}^{\infty} \frac{\mathrm{d}\lambda}{2\pi} A^{(\sigma)}(\lambda)\mathrm{e}^{-\mathrm{i}\lambda t}|\lambda, \sigma\rangle. \tag{5.4}$$

Let $\Phi_a(t)$ be the probability amplitude for the ath atom to be excited. Then

$$\Phi_1(t) \equiv \langle 0|\sigma_1^-|\Phi(t)\rangle = \int_{-\infty}^{\infty} \frac{\mathrm{d}\lambda}{2\pi} \mathrm{e}^{-\mathrm{i}\lambda t} \left(A^{(\mathrm{s})}(\lambda)\xi^{(\mathrm{s})}(\lambda) + A^{(\mathrm{a})}(\lambda)\xi^{(\mathrm{a})}(\lambda)\right) \tag{5.5}$$

$$\Phi_2(t) \equiv \langle 0|\sigma_2^-|\Phi(t)\rangle = \int_{-\infty}^{\infty} \frac{\mathrm{d}\lambda}{2\pi} \mathrm{e}^{-\mathrm{i}\lambda t} \left(A^{(\mathrm{s})}(\lambda)\xi^{(\mathrm{s})}(\lambda) - A^{(\mathrm{a})}(\lambda)\xi^{(\mathrm{a})}(\lambda)\right) \tag{5.6}$$

which describe the temporal behaviour of each atom's excitation. Using the explicit expressions (4.15), (4.20), and (5.3), one obtains

$$\Phi_1(t) = -\int_{-\infty}^{\infty} \frac{\mathrm{d}\lambda}{4\pi\mathrm{i}} \left(\frac{1}{r_+(\lambda)} + \frac{1}{r_-(\lambda)}\right) \mathrm{e}^{-\mathrm{i}\lambda t} \tag{5.7}$$

$$\Phi_2(t) = -\int_{-\infty}^{\infty} \frac{d\lambda}{4\pi\mathrm{i}} \left(\frac{1}{r_+(\lambda)} - \frac{1}{r_-(\lambda)}\right) \mathrm{e}^{-\mathrm{i}\lambda t} = \gamma \int_{-\infty}^{\infty} \frac{\mathrm{d}\lambda}{2\pi} \frac{\mathrm{e}^{\mathrm{i}\lambda(l-t)}}{r_+(\lambda)r_-(\lambda)} \tag{5.8}$$

where

$$r_+(\lambda) = \lambda - \omega_{12}^{(\mathrm{s})} + \mathrm{i}\gamma(1 + \mathrm{e}^{\mathrm{i}\lambda l}) \qquad r_-(\lambda) = \lambda - \omega_{12}^{(\mathrm{a})} + \mathrm{i}\gamma(1 - \mathrm{e}^{\mathrm{i}\lambda l}) \tag{5.9}$$

and $l = |x_1 - x_2|\rangle 0$ is the interatomic separation.

Now, if one simply considers the imaginary part of (5.9), it is very easy to show that zeros of $r_\pm(\lambda)$ never occur in the upper half λ-plane. This fact leads immediately to the vanishing of the wavefunction of the initially unexcited atom for $t < l$, $\Phi_2(t < l) = 0$. Thus causality is preserved. It is easily to see also that the frequency splitting between the symmetric and antisymmetric eigenstates, $\Delta(l) = \omega_{12}^{(\mathrm{s})}(l) - \omega_{12}^{(\mathrm{s})}(l)$, plays no role in causal behaviour of the system. Therefore, to avoid more complicated expressions, in the following calculations we confine ourselves to the case of long interatomic separations when the frequency splitting can be neglected.

Furthermore, it is rather simple to continue and evaluate in an explicit form (5.7) and (5.8), which we shall need in the next section. In the dimensionless variables

$$\lambda' = l\lambda \qquad \gamma' = l\gamma \qquad t' = \frac{t}{l} \tag{5.10}$$

the integrals from (5.7) and (5.8) take the form

$$I_{\pm}(t) = -\int_{-\infty}^{\infty} \frac{\mathrm{d}\lambda}{4\pi\mathrm{i}} \frac{\mathrm{e}^{-\mathrm{i}\lambda t}}{r_{\pm}(\lambda)} \tag{5.11}$$

$$r_{\pm}(\lambda) = \lambda + \mathrm{i}\gamma(1 \pm \mathrm{e}^{\mathrm{i}(\lambda+\phi_0)}) \tag{5.12}$$

where $\phi_0 = \omega_{12} l$ and all primes and the trivial factor $\exp(-\mathrm{i}\omega_{12}t)$ in front of integral have been omitted.

The poles of (5.11) are transcendental, but we may still evaluate (5.11) by using the following trick. Introducing the new variable

$$\lambda = 2\pi n + \mu$$

where $\mu \in (0, 2\pi)$, $n = 0, \pm 1, \ldots$, one can represent (5.11) as follows:

$$I_{\pm}(t) = -\int_0^{2\pi} \frac{\mathrm{d}\mu}{4\pi\mathrm{i}} \mathrm{e}^{-\mathrm{i}\mu t} \sum_{n=-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}2\pi n t}}{2\pi n + r_{\pm}(\mu)}. \tag{5.13}$$

The sum

$$\psi_{\pm}(t) = \sum_{n=-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}2\pi n t}}{2\pi n + r_{\pm}(\mu)} \tag{5.14}$$

can be treated as the Fourier series of the periodic function $\psi_{\pm}(t+T) = \psi_{\pm}(t)$:

$$\psi_{\pm}(t) = \sum_{n=-\infty}^{\infty} \psi_{\pm}(\omega_n)\mathrm{e}^{-\mathrm{i}\omega_n t} \tag{5.15}$$

$$\psi_{\pm}(\omega_n) = \frac{1}{T}\int_0^T \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega_n t}\psi_{\pm}(t) \tag{5.16}$$

where $\omega_n = 2\pi n/T$. In our case the period is $T = 1$ (or $T = l^{-1}$ in dimension variables), and we easily find:

$$\psi_{\pm}(\mu, t) = -\frac{\mathrm{i}}{1 - \exp[\mathrm{i}r_{\pm}(\mu)]} \exp[\mathrm{i}r_{\pm}(\mu)t] \qquad t \in (0, 1) \tag{5.17}$$

$$\psi_{\pm}(\mu, t+n) = \psi_{\pm}(\mu, t) \tag{5.18}$$

and

$$I_{\pm}(t) = -\int_0^{2\pi} \frac{\mathrm{d}\mu}{4\pi\mathrm{i}} \mathrm{e}^{-\mathrm{i}\mu t}\psi_{\pm}(\mu, t). \tag{5.19}$$

Due to the periodicity of the function $\psi_{\pm}(\mu, t)$, the temporal axis $0 \leqslant t < \infty$ conveniently divides into an infinite set of temporal zones. Now we can represent the temporal variable t as

$$t = \tau + n \tag{5.20}$$

where $\tau \in (0, 1)$ and $n = 0, 1, \ldots$. Then, for the nth temporal zone the function $I_{\pm}^{(n)}(\tau) \equiv I_{\pm}(\tau + n)$ is given by

$$I_{\pm}^{(n)}(\tau) = \int_0^{2\pi} \frac{\mathrm{d}\mu}{4\pi} \frac{\mathrm{e}^{-\mathrm{i}\mu n}}{1 - \exp[\mathrm{i}r_{\pm}(\mu)]} \exp[\mathrm{i}(r_{\pm}(\mu) - \mu)\tau] \tag{5.21}$$

or, in terms of the variable $z = e^{i(\phi_0+\mu)}$,

$$I_{\pm}^{(n)}(\tau) = e^{in\phi_0}e^{-\gamma\tau}\oint \frac{dz}{4\pi i}\frac{1}{z^{n+1}}F_{\pm}(z) \tag{5.22}$$

where

$$F_{+}(z) = \frac{e^{-\gamma\tau z}}{1 - ze^{-\gamma z}e^{-(i\phi_0+\gamma)}} \qquad F_{-}(z) = \frac{e^{\gamma\tau z}}{1 - ze^{\gamma z}e^{-(i\phi_0+\gamma)}} \tag{5.23}$$

and the integral is taken in the positive direction along the circle of unit radius with the centre at the point $z = 0$.

It is easy to show the functions $F_{\pm}(z)$ have no pole inside the integration contour. Thus one obtains

$$I_{\pm}^{(n)}(\tau) = e^{-i\omega_{12}t}e^{in\phi_0}e^{-\gamma\tau}\frac{1}{2}\frac{1}{n!}\left[\frac{d^n}{dz^n}F_{\pm}(z)\right]_{|z=0} \tag{5.24}$$

$$F_{+}(z) = \frac{e^{-\gamma\tau z}}{1 - ze^{-\gamma l z}e^{-(i\phi_0+\gamma l)}} \qquad F_{-}(z) = \frac{e^{\gamma\tau z}}{1 - ze^{\gamma l z}e^{-(i\phi_0+\gamma l)}} \tag{5.25}$$

upon restoring the parameter l. Now $\tau \in (0, l)$.

Thus, we finally find for the atomic wavefunctions:

$$\begin{aligned}\Phi_1^{(n)}(\tau) &= e^{-i\omega_{12}t}e^{in\phi_0}e^{-\gamma\tau}\frac{1}{2}\frac{1}{n!}\left(\frac{d^n}{dz^n}[F_{+}(z) + F_{-}(z)]\right)_{z=0} \\ &= e^{-i(\omega_{12}-i\gamma)t}\frac{1}{2}\sum_{m=0}^{n}\frac{(n-m)!}{n!}\gamma^m\left[(ml-t)^m + (t-ml)^m\right]e^{m(\gamma 1+i\phi_0)}\end{aligned} \tag{5.26}$$

$$\begin{aligned}\Phi_2^{(n)}(\tau) &= e^{-i\omega_{12}t}e^{in\phi_0}e^{-\gamma\tau}\frac{1}{2}\frac{1}{n!}\left(\frac{d^n}{dz^n}[F_{+}(z) - F_{-}(z)]\right)_{z=0} \\ &= e^{-i(\omega_{12}-i\gamma)t}\frac{1}{2}\sum_{m=0}^{n}\frac{(n-m)!}{n!}\gamma^m\left[(ml-t)^m - (t-ml)^m\right]e^{m(\gamma l+i\phi_0)}\end{aligned} \tag{5.27}$$

where in the second equalities, $t = \tau + nl$. It is easy to see that the functions $\Phi_1^{(n)}$ and $\Phi_2^{(n)}$ contain only even ($m = 2k$) or odd ($m = 2k+1$) terms of the sum, respectively. Therefore, the functional form of $\Phi_1^{(n)}$ does not change at the odd values of $n = 2k+1$. The same is true for the function $\Phi_2^{(n)}$ except at the even values of $n = 2k$.

In the limiting case of very large interatomic distances $l \gg \gamma^{-1}$, one needs to only consider the first few temporal zones. Then, we have:

$$\Phi_1(t) = e^{-i\omega_{12}t}e^{-\gamma t}\begin{cases}1 & 0 \leqslant t < 2l \\ 1 - \frac{1}{2}\gamma^2(t-2l)^2e^{2(i\omega_{12}l+\gamma l)} & 2l \leqslant t < 4l\end{cases} \tag{5.28}$$

$$\Phi_2(t) = e^{-i\omega_{12}t}e^{-\gamma t}\begin{cases}0 & 0 \leqslant t < l \\ -\frac{1}{2}\gamma(t-l)e^{(i\omega_{12}l+\gamma l)} & l \leqslant t < 3l\end{cases} \tag{5.29}$$

and so on. The 'retardation' effect is obviously taking place at $t = 2l$ and repeats with a period of $T = 2l$. This is the minimum time it takes for the first atom to transmit a photon to the second atom and then for the second atom to return the photon to the first atom.

6. Spectral density of radiation

Let us now take our solution (5.4) and determine the spectral density of the emitted photons.

In the limit $t \to \infty$, the contribution of the atomic variables to the state $|\Phi(t)\rangle$ (4.4) vanishes, (because eventually it decays), and we find for the asymptotic state

$$|\Phi(t \to \infty)\rangle = \exp(-\mathrm{i}H_{\mathrm{F}}t)|\mathrm{Out}\rangle \tag{6.1}$$

where H_{F} is the Hamiltonian of the free field,

$$H_{\mathrm{F}} = -\mathrm{i}\int_{-\infty}^{\infty} \mathrm{d}x \left(\epsilon_{\mathrm{R}}^{+}(x)\frac{\partial}{\partial x}\epsilon_{\mathrm{R}}(x) - \epsilon_{\mathrm{L}}^{+}(x)\frac{\partial}{\partial x}\epsilon_{\mathrm{L}}(x)\right) \tag{6.2}$$

and the out-state containing the radiated field is given by

$$|\mathrm{Out}\rangle = \int_{-\infty}^{\infty} \mathrm{d}x[\psi_{\mathrm{R}}(x)\epsilon_{\mathrm{R}}^{+}(x) + \psi_{\mathrm{L}}(x)\epsilon_{\mathrm{L}}^{+}(x)]|0\rangle \tag{6.3}$$

where the photon wavefunctions are

$$\psi_{\mathrm{R}}(x) = \frac{\sqrt{\gamma}}{2}\int_{-\infty}^{\infty}\frac{\mathrm{d}\lambda}{2\pi}\{R_{+}(\lambda)\mathrm{e}^{-\mathrm{i}\lambda(x-x_1)} + R_{-}(\lambda)\mathrm{e}^{-\mathrm{i}\lambda(x-x_2)}\} \tag{6.4}$$

$$\psi_{\mathrm{L}}(x) = \frac{\sqrt{\gamma}}{2}\int_{-\infty}^{\infty}\frac{\mathrm{d}\lambda}{2\pi}\{R_{+}(\lambda)\mathrm{e}^{\mathrm{i}\lambda(x-x_1)} + R_{-}(\lambda)\mathrm{e}^{\mathrm{i}\lambda(x-x_2)}\} \tag{6.5}$$

and

$$R_{\pm}(\lambda) = \frac{1}{r_{+}(\lambda)} \pm \frac{1}{r_{-}(\lambda)}. \tag{6.6}$$

The spectral density of radiation propagating in the positive and negative directions of the x-axis is determined by the obvious expressions

$$G_{\mathrm{R}}(\omega) = \psi_{\mathrm{R}}^{*}(\omega)\psi_{\mathrm{R}}(\omega) \qquad G_{\mathrm{L}}(\omega) = \psi_{\mathrm{L}}^{*}(\omega)\psi_{\mathrm{L}}(\omega) \tag{6.7}$$

where $\psi_{\mathrm{R,L}}(\omega)$ are the Fourier transform of the asymptotic wavefunction,

$$\psi_{\mathrm{R}}(\omega) = \int_{-\infty}^{\infty}\mathrm{d}x\,\psi_{\mathrm{R}}(x)\mathrm{e}^{-\mathrm{i}\omega x} = \frac{\sqrt{\gamma}}{2}(R_{+}(\omega)\mathrm{e}^{-\mathrm{i}\omega x_1} + R_{-}(\omega)\mathrm{e}^{-\mathrm{i}\omega x_2}) \tag{6.8}$$

$$\psi_{\mathrm{L}}(\omega) = \int_{-\infty}^{\infty}\mathrm{d}x\,\psi_{\mathrm{L}}(x)\mathrm{e}^{\mathrm{i}\omega x} = \frac{\sqrt{\gamma}}{2}(R_{+}(\omega)\mathrm{e}^{\mathrm{i}\omega x_1} + R_{-}(\omega)\mathrm{e}^{\mathrm{i}\omega x_2}). \tag{6.9}$$

Then, we find

$$G_{\mathrm{R,L}}(\omega) = \frac{1}{2}\left(\frac{\Gamma^{(\mathrm{s})}}{(\omega-\Omega^{(\mathrm{s})})^2 + (\Gamma^{(\mathrm{s})})^2} + \frac{\Gamma^{(\mathrm{a})}}{(\omega-\Omega^{(\mathrm{a})})^2 + (\Gamma^{(\mathrm{a})})^2}\right)$$
$$\pm\gamma\sin[\omega(x_1-x_2)]\frac{\Gamma^{(\mathrm{s})}(\omega-\Gamma^{(\mathrm{a})}) - \Gamma^{(\mathrm{a})}(\omega-\Gamma^{(\mathrm{s})})}{[(\omega-\Omega^{(\mathrm{s})})^2 + (\Gamma^{(\mathrm{s})})^2][(\omega-\Omega^{(\mathrm{a})})^2 + (\Gamma^{(\mathrm{a})})^2]} \tag{6.10}$$

where the positive sign in front of the third term corresponds to G_{R}, and the negative sign to G_{L}. If two detectors collect the radiation emitted in both directions of the x-axis, the measured spectral density will be given by

$$G(\omega) = G_{\mathrm{R}} + G_{\mathrm{L}} = \frac{\Gamma^{(\mathrm{s})}}{(\omega-\Omega^{(\mathrm{s})})^2 + (\Gamma^{(\mathrm{s})})^2} + \frac{\Gamma^{(\mathrm{a})}}{(\omega-\Omega^{(\mathrm{a})})^2 + (\Gamma^{(\mathrm{a})})^2}. \tag{6.11}$$

The expression (6.10) is a periodic function of the interatomic separation $l = |x_1 - x_2|$, and, hence, the contribution of atom–atom correlations to the spectral density of radiation does not vanish at any l. Then, the spectral density of radiation consists of two Lorentz lines of width

$$\Gamma^{(\mathrm{s})} = \gamma(1 + \cos[\omega_{12}(x_1 - x_2)]) \qquad \Gamma^{(\mathrm{a})} = \gamma(1 - \cos[\omega_{12}(x_1 - x_2)])$$

located at the frequencies

$$\Omega^{(s)} = \omega_{12} + \gamma \sin(\omega_{12}|x_1 - x_2|) \qquad \Omega^{(a)} = \omega_{12} - \gamma \sin(\omega_{12}|x_1 - x_2|)$$

respectively, and the dynamics of our initial problem can be treated as an independent exponential spontaneous decay of symmetric and antisymmetric states of the two-atom system, in which the interatomic correlations are generated by virtual photon exchanges during a time shorter than the spontaneous relaxation time γ^{-1} [6, 8].

7. Many-atom model

In this section of the article we derive the equations describing the dynamics of a one-particle excitation in a system of M identical atoms located at the points $\{x_a, a = 1, \ldots, M\}$. The Hamiltonian of the model is obviously to be

$$H = \omega_{12} \sum_{a=1}^{M} (\sigma_a^z + \tfrac{1}{2}) + \int_{-\infty}^{\infty} \mathrm{d}x \Big\{ -\mathrm{i}\epsilon_{\mathrm{R}}^{+}(x) \frac{\partial}{\partial x} \epsilon_{\mathrm{R}}(x) + \mathrm{i}\epsilon_{\mathrm{L}}^{+}(x) \frac{\partial}{\partial x} \epsilon_{\mathrm{L}}(x) + \sqrt{\gamma} \sum_{a=1}^{M} \delta(x - x_a)[(\epsilon_{\mathrm{R}}(x) + \epsilon_{\mathrm{L}}(x))\sigma_a^{+} + (\epsilon_{\mathrm{R}}^{+}(x) + \epsilon_{\mathrm{L}}(x))\sigma_a^{-}] \Big\} \tag{7.1}$$

where we also confine ourselves to the case of long interatomic separations.

The one-particle eigenstates are given by

$$|\lambda\rangle = \Big[\sum_{a=1}^{M} \xi_a \sigma_a^{+} + \int_{-\infty}^{\infty} \mathrm{d}x (f_{\mathrm{R}}(x)\epsilon_{\mathrm{R}}^{+}(x) + f_{\mathrm{L}}(x)\epsilon_{\mathrm{L}}^{+}(x)) \Big] |0\rangle \tag{7.2}$$

where the wavefunctions are determined from the Schrödinger equation which in our case takes the following form:

$$\Big(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x} + \lambda \Big) f_{\mathrm{R}}(x, \lambda) = \sqrt{\gamma} \sum_{a=1}^{M} \xi_a(\lambda) \delta(x - x_a) \tag{7.3}$$

$$\Big(-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x} + \lambda \Big) f_{\mathrm{L}}(x, \lambda) = \sqrt{\gamma} \sum_{a=1}^{M} \xi_a(\lambda) \delta(x - x_a) \tag{7.4}$$

$$(\lambda - \omega_{12}) \xi_a(\lambda) = \sqrt{\gamma} \int_{-\infty}^{\infty} \mathrm{d}x\, \delta(x - x_a)(f_{\mathrm{R}}(x, \lambda) + f_{\mathrm{L}}(x, \lambda)). \tag{7.5}$$

The general solution of (7.3) and (7.4) is

$$f_{\mathrm{R}}(x, \lambda) = C_{\mathrm{R}}(\lambda) \mathrm{e}^{\mathrm{i}\lambda x} - \mathrm{i}\sqrt{\gamma} \sum_{a=1}^{M} \xi_a(\lambda) \mathrm{e}^{\mathrm{i}\lambda(x - x_a)} \theta(x - x_a) \tag{7.6}$$

$$f_{\mathrm{L}}(x, \lambda) = C_{\mathrm{L}}(\lambda) \mathrm{e}^{-\mathrm{i}\lambda x} - \mathrm{i}\sqrt{\gamma} \sum_{a=1}^{M} \xi_a(\lambda) \mathrm{e}^{-\mathrm{i}\lambda(x - x_a)} \theta(x_a - x) \tag{7.7}$$

(where $C_{\mathrm{R,L}}$ are arbitrary constants) which when substituted into (7.5) yields the following set of linear algebraic equations for the atomic wavefunctions:

$$(\lambda - \omega_{12}) \xi_a + \mathrm{i}\gamma \sum_{b=1}^{M} \mathrm{e}^{\mathrm{i}\lambda|x_a - x_b|} \xi_b = C_{\mathrm{R}} \mathrm{e}^{\mathrm{i}\lambda x_a} + C_{\mathrm{L}} \mathrm{e}^{-\mathrm{i}\lambda x_a}. \tag{7.8}$$

We will now generalize the preceding to the case of a continuous resonance medium. Then equation (7.8) takes the form:

$$(\lambda - \omega_{12})\xi(x,\lambda) + \mathrm{i}\gamma \int_0^L \mathrm{d}y\, \rho(y)\mathrm{e}^{\mathrm{i}\lambda|x-y|}\xi(y,\lambda) = C_\mathrm{R}\mathrm{e}^{\mathrm{i}\lambda x} + C_\mathrm{L}\mathrm{e}^{-\mathrm{i}\lambda x} \qquad (7.9)$$

where $\rho(x)$ is the linear number density of the atoms,

$$\int_0^L \mathrm{d}x\, \rho(x) = M$$

and L is the medium's length. The expressions (7.6) and (7.7) for the photon wavefunctions become respectively

$$f_\mathrm{R}(x,\lambda) = C_\mathrm{R}(\lambda)\mathrm{e}^{\mathrm{i}\lambda x} - \mathrm{i}\sqrt{\gamma}\int_0^x \mathrm{d}y\, \rho(y)\xi(y,\lambda)\mathrm{e}^{\mathrm{i}\lambda(x-y)} \qquad (7.10)$$

$$f_\mathrm{L}(x,\lambda) = C_\mathrm{L}(\lambda)\mathrm{e}^{-\mathrm{i}\lambda x} - \mathrm{i}\sqrt{\gamma}\int_x^L \mathrm{d}y\, \rho(y)\xi(y,\lambda)\mathrm{e}^{-\mathrm{i}\lambda(x-y)}. \qquad (7.11)$$

Now consider what happens to the reflection and transmission of an incident light from a resonance medium with the density of the number of atoms, such as

$$\rho(x) = \frac{M}{L} + \delta\rho(x) \qquad (7.12)$$

where $\delta\rho(x)$ is a random function and represents fluctuations. It satisfies

$$\langle\delta\rho(x)\rangle = 0 \qquad \langle\delta\rho(x)\delta\rho(y)\rangle = \beta\delta(x-y). \qquad (7.13)$$

Here the symbol $\langle\ldots\rangle$ stands for averaging over disorder. Applying the operator $(\mathrm{d}^2/\mathrm{d}x^2 + \lambda^2)$ to both sides of (6.9), one can rewrite this integral equation in the form of the Schrödinger equation with a partially random potential,

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{2\lambda\gamma}{\lambda - \omega_{12}}\rho(x)\right)\xi(x,\lambda) = \lambda^2\xi(x,\lambda). \qquad (7.14)$$

It is well known that all the states of this equation are localized [**?**] (or quasi-localized for a finite size medium) for any arbitrarily small parameter β, characterizing the strength of the disorder. This leads to a non-trivial evolution of the reflected light [19, 20].

In other words, there will be a 'light localization' wherein any photon will bounce back and forth, on average, leading it to be more likely found inside some localized region. Now, as one can see from (7.14), there is a resonant denominator. Thus, if there are fluctuations in $\delta\rho(x)$, with wavelengths matching the wavelengths of the localized photons, then a considerable enhancement of the localized photon density could be expected.

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